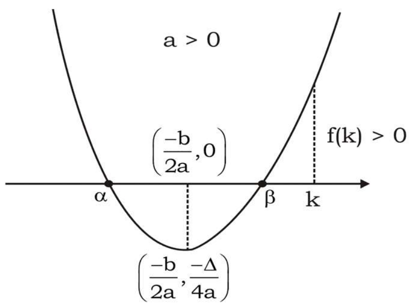
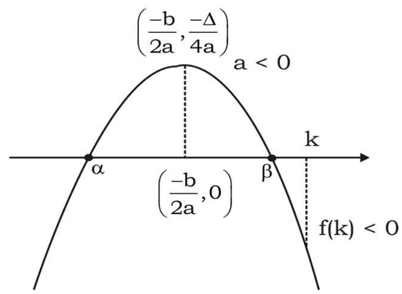
Various cases of location of roots:

Let $f(x) = ax^2 + bx + c, a \neq 0$ and a, b, c $\in R$, and α, β are roots of $f(x) = 0$ where $\alpha \leq \beta$

Suppose $k, k_1, k_2 \in R$ and $k_1 < k_2$, then remember the following.

CASE – I: Condition for a number k if both roots of $f(x) = 0$ are less than k.

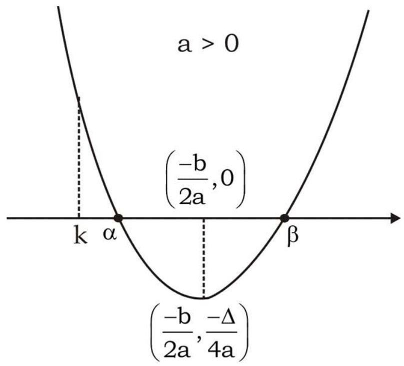


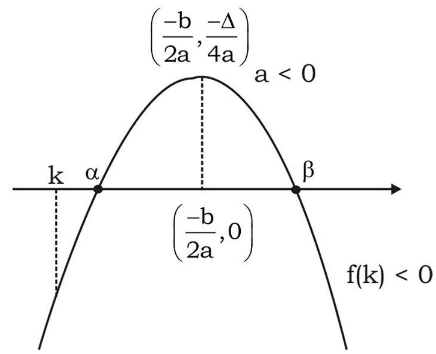


i) $\Delta \geq 0$ ii) $af(k) > 0$ iii) $k > \dfrac{-b}{2a}$

Intersection of (i), (ii) and (iii) gives the result.

CASE – II: Condition for both roots of $f(x)=0$ are greater than k.

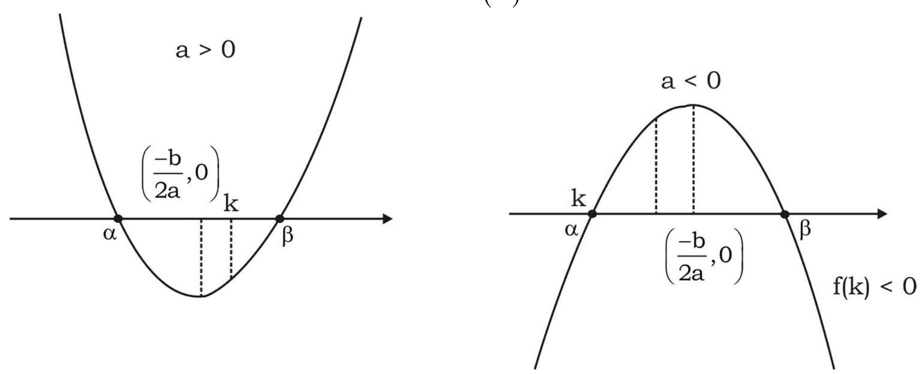




i) $\Delta \geq 0$ ii) $af(k) > 0$ iii) $k < \frac{-b}{2a}$

Intersection of (i), (ii) and (iii) gives the result.

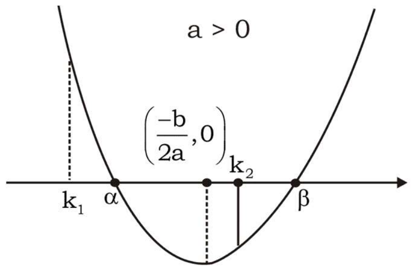
CASE – III: If k lies between the roots of $f(x)=0$

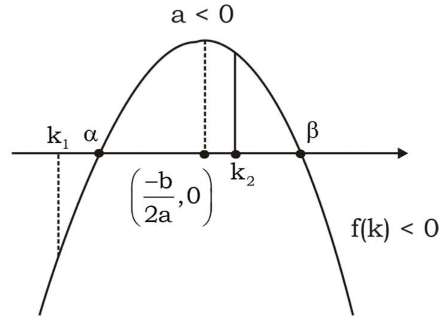


i) $\Delta > 0$ ii) $af(k) < 0$

Intersection of (i) and (ii) gives the result.

CASE – IV: Condition for the numbers k_1 and k_2 if exactly one root of $f(x)=0$ lies in the interval (k_1, k_2).

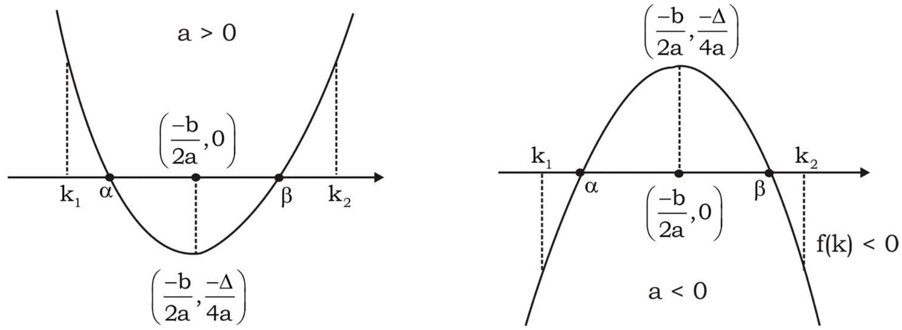




i) $\Delta > 0$ ii) $f(k_1)f(k_2) < 0$

Intersection of (i) and (ii) gives the result.

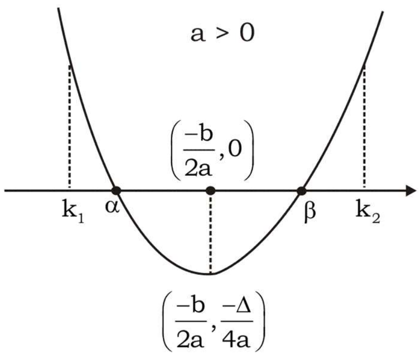
CASE – V: Condition for numbers k_1 and k_2 if both roots of $f(x)=0$ are confined between k_1 and k_2.

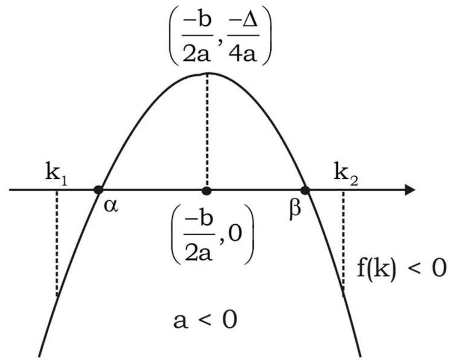


i) $\Delta \ge 0$ ii) $af(k_1)>0, af(k_2)>0$ iii) $k_1 < \frac{-b}{2a} < k_2$ where $\alpha \le \beta$ and $k_1 < k_2$

Intersection of (i), (ii) and (iii) gives the result.

CASE – VI: Condition for numbers k_1 and k_2 if k_1 and k_2 lie between the roots $f(x)=0$





i) $\Delta > 0$ ii) $af(k_1) < 0$, $af(k_2) < 0$ where $k_1 < k_2, \alpha < \beta$.

Intersection of (i) and (ii) gives the result.

Let $x^2 - (m-3)x + m = 0 \, (m \in R)$ be a quadratic equation. Find the values of m for which the roots are

i) Real and distinct

ii) Equal

iii) Not real

iv) Opposite in sign

v) Equal in magnitude but opposite in sign

vi) Positive

vii) Negative

viii) Such that at least one is positive

ix) One root is smaller than 2 and the other root is greater than 2

x) Both the roots are greater than 2

xi) Both the roots are smaller than 2

xii) Exactly one root lies in the interval $(1, 2)$

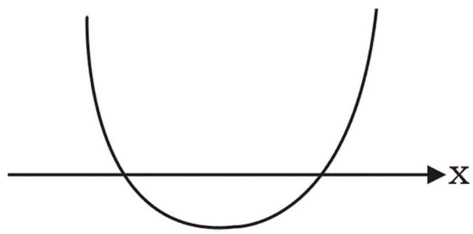
xiii) Both the roots lie in the interval $(1, 2)$

xiv) At least one root lies in the interval $(1, 2)$

xv) One root is greater than 2 and the other root is smaller than1

Sol: Let $f(x) = x^2 - (m-3)x + m = 0$

i)



Both the roots are real and distinct. So,

$D > 0$

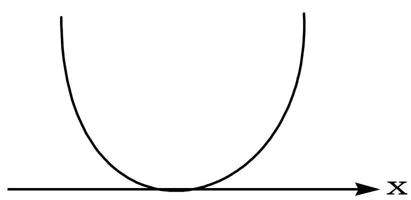
$\Rightarrow (m-3)^2 - 4m > 0$

$\Rightarrow m^2 - 10m + 9 > 0$

$\Rightarrow (m-1)(m-9) > 0$

$\Rightarrow m \in (-\infty, 1) \cup (9, \infty)$

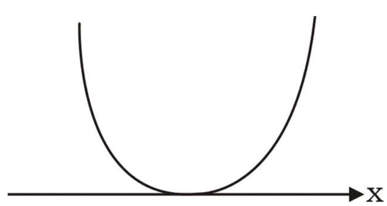
ii)



Both the roots are equal. So,

$D = 0 \Rightarrow m = 9$ or $m = 1$

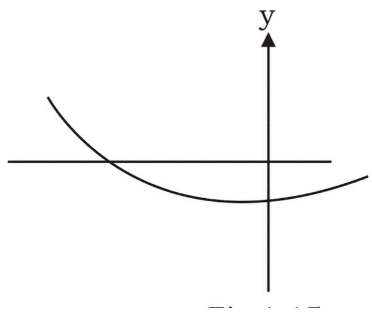
iii)



Both the roots are imaginary. So,

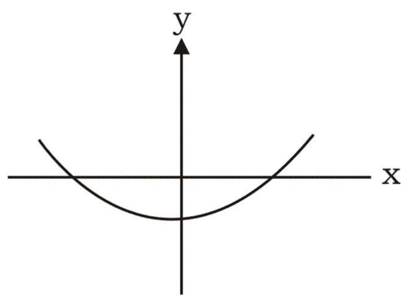
$D < 0 \Rightarrow (m-1)(m-9) < 0$ $\quad m \in (1, 9)$

iv)



Both the roots are opposite in sign. Hence, the product of roots is negative.
So, $m < 0 \Rightarrow m \in (-\infty, 0)$

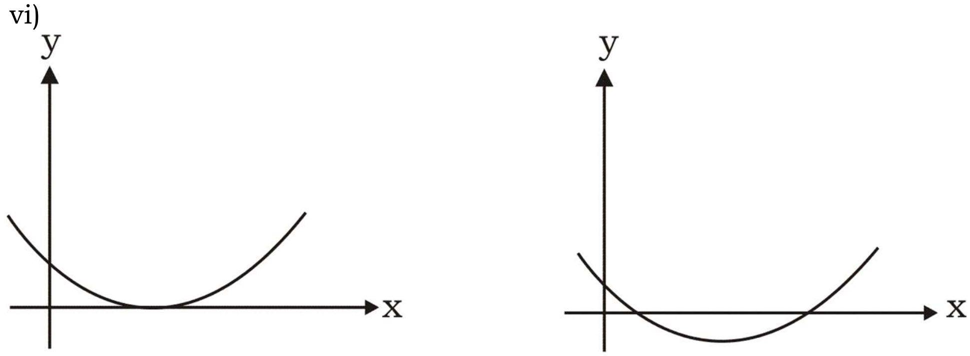
v)



Roots are equal in magnitude but opposite in sign. Hence, sum of roots is zero as well as $D \geq 0$. So, $m \in (-\infty, 1) \cup (9, \infty)$ and $m - 3 = 0$ i.e., $m = 3$

As no such m exists, so $m \in \phi$

vi)

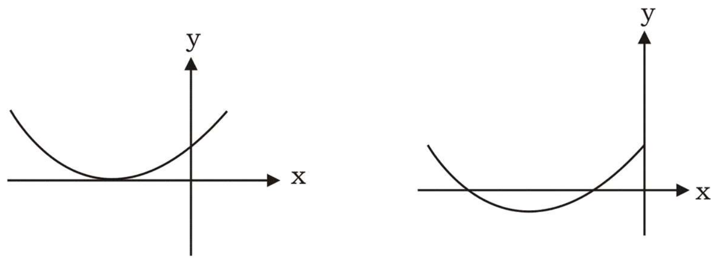


Both the roots are positive. Hence, $D \geq 0$ and both the sum and the product of roots are positive. So,

$m - 3 > 0, m > 0$ and $m \in (-\infty, 1] \cup [9, \infty)$

$m \in [9, \infty)$

vii)

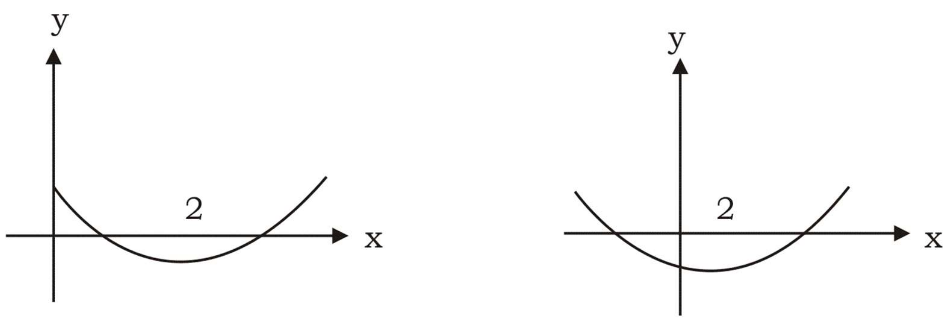


Both the roots are negative. Hence, $D \geq 0$, and sum is negative but product is positive. So,

$m-3<0, m>0$ $m \in (-\infty,1] \cup [9,\infty)$ $\Rightarrow$ $m \in (0,1]$

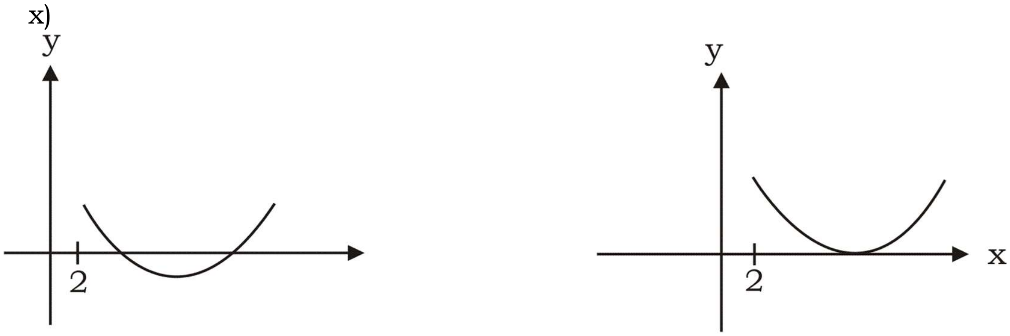
viii) At least one root is positive. Hence, either one root is positive or both roots are positive. So, $m \in (-\infty,0) \cup [9,\infty)$

ix)



One root is smaller than 2 and the other root is greater than 2, i.e., 2 lies between the roots. So, $f(2)<0$ $\Rightarrow$ $4-2(m-3)+m<0$ $\Rightarrow$ $m>10$

x)

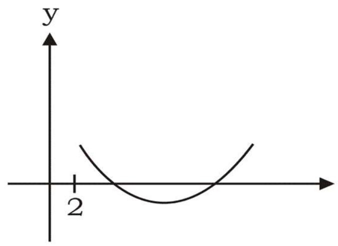


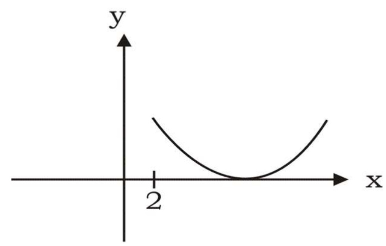
Both the roots are greater than 2. So, $f(2)>0,\ D \geq 0,\ -\dfrac{b}{2a}>2$

$\Rightarrow$ $m<10$ and $m \in (-\infty,1] \cup [9,\infty)$ and $m-3>4$

$\Rightarrow$ $m \in [9,10)$

xi)

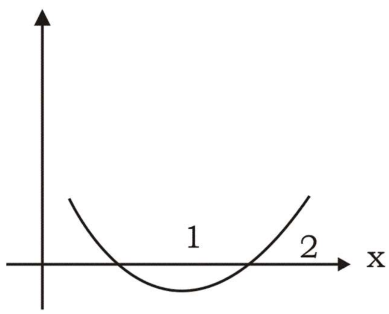


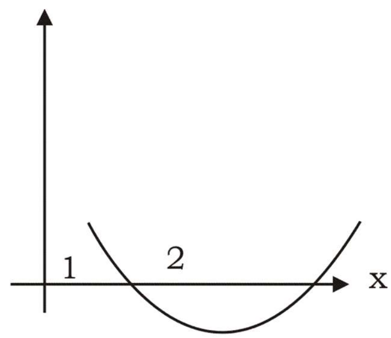


Both the roots are smaller than 2. So, $f(2) > 0, D \ge 0, -\frac{b}{2a} < 2$

$\Rightarrow \ m \in (-\infty, 1]$

xii)





Exactly one root lies in $(1, 2)$. So, $f(1)f(2) < 0$

$\Rightarrow \ 4(10 - m) < 0$

$\Rightarrow \ m \in (10, \infty)$

xiii) Both the roots lie in the interval $(1, 2)$. Then,

$D \ge 0 \Rightarrow (m-1)(m-9) \ge 0 \Rightarrow \ m \le 1 \text{ or } m \ge 9$ ...(1)

Also,

$f(1) > 0 \text{ and } f(2) > 0 \Rightarrow \ 10 > m$ ...(2)

And

$1 < -\frac{b}{2a} < 2 \Rightarrow \ 5 < m < 7$ ...(3)

Thus, no such m exists.

xiv)

CASE – I: Exactly one root lies in $(1, 2)$. So,

$f(1)f(2) < 0 \ \Rightarrow \ m > 10$

CASE – II: Both the roots lie in $(1, 2)$. So, from (xiii) $m \in \phi$. Hence, $m \in (10, \infty)$

.

xv) For one root greater than 2 and the other root smaller than 1,

$f(1) < 0 \quad ...(1)$

$f(2) < 0 \quad ...(2)$

From (1), $f(1) < 0$, but $f(1) = 4$, which is not possible. Thus, no such m exists.